J. Fluid Mech. (1970), *vol.* 43, *part* 4, *pp.* 767–770
Printed in Great Britain

Wave drag caused by a swirling flow through a convergent-divergent nozzle

By CHUEN-YEN CHOW AND YING-CHUNG LAI

Department of Aerospace Engineering Sciences
University of Colorado, Boulder, Colorado

(Received 15 January 1970)

The wave drag coefficient is computed approximately for a nozzle containing a swirling flow as a function of R_o^{-1}, the inverse of the Rossby number. When $R_o^{-1} < \lambda_1$ and $R_o^{-1} = \lambda_n$, where λ_n denotes the nth zero of the Bessel function J_1, there is no wave in the flow and the wave drag is zero. The drag coefficient is found to be sub-divided into different regions between $R_o^{-1} = \lambda_n$ and λ_{n+1}, with $n = 1, 2, 3, \ldots$. When each λ_n is exceeded, the drag coefficient jumps from zero to a value which is one order higher than its values in the previous region (except in the case $n = 1$), and then decreases to zero as R_o^{-1} increases toward λ_{n+1}. Very high wave drag can be expected in flows of large swirl ratios.

1. Introduction

In a previous work (Chow 1969) a problem was studied analytically concerning a swirling flow of an inviscid, incompressible fluid through an axisymmetric convergent-divergent nozzle. It was found that if the angular velocity of the flow is small compared with the axial velocity, the flow patterns up- and downstream from a contraction are symmetric. But when the swirl ratio exceeds a certain critical value, the flow becomes non-symmetric and waves appear downstream from the throat. These internal waves persist in the flow at higher swirl ratios except under special conditions when the flow becomes blocked by the contraction.

Associated with the downstream waves, there must be a drag acting on the nozzle. Fraenkel (1956) studied this problem but did not go any further after an expression for the drag was derived. The expression was in the form of a series, whose physical meaning can hardly be interpreted unless some numerical examples are given.

In the present work we will compute the wave drag of a nozzle of specified shape at different swirl ratios. Since the contour of the nozzle is to be constructed through an inverse method, it deviates considerably when the swirl ratio is changed. The curve for the drag so obtained is therefore an approximated one for a given nozzle.

2. The analytical model

The same mathematical model used by Chow will be adopted here, which is described briefly as follows. Far upstream from the contraction, the tube has a constant radius R and the fluid, rotating at a constant angular speed ω, has a uniform axial velocity W along the positive z axis which coincides with the centre-line of the tube. The contraction is generated by superimposing on this uniform flow a distribution of ring vortices of strength $f(r)$ over the range $r_1 \leqslant r \leqslant R$ at the section $z = 0$. The resulting stream function, whose value and r derivatives are continuous at $z = 0$, is separated into two parts. For the flow upstream from the throat in the region $z \leqslant 0$,

$$\psi_- = WR^2\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 + \sum_{N+1}^{\infty} a_n e^{p_n z}\frac{r}{R}J_1\left(\lambda_n\frac{r}{R}\right)\right], \tag{1}$$

and for that downstream in the region $z \geqslant 0$,

$$\psi_+ = WR^2\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 + \sum_{n=1}^{N} b_n\frac{r}{R}J_1\left(\lambda_n\frac{r}{R}\right)\sin k_n z + \sum_{N+1}^{\infty} a_n e^{-p_n z}\frac{r}{R}J_1\left(\lambda_n\frac{r}{R}\right)\right]. \tag{2}$$

p_n and k_n are defined respectively by the relations

$$p_n R = (\lambda_n^2 - R_o^{-2})^{\frac{1}{2}} \tag{3}$$

and

$$k_n R = (R_o^{-2} - \lambda_n^2)^{\frac{1}{2}} \tag{4}$$

in which R_o ($= W/2R\omega$) is the Rossby number whose inverse characterizes the swirl ratio, and λ_n is the nth zero of the Bessel function J_1. The integer N is determined by comparing the magnitude of R_o with the λ_n's through the relation

$$\lambda_N < |R_o^{-1}| \leqslant \lambda_{N+1}. \tag{5}$$

In the case $N = 0$, all the terms containing sine functions are omitted. The expressions for a_n and b_n were shown in the previous work corresponding to the chosen function

$$f(r) = Q\frac{r}{R}J_1\left(\lambda_1\frac{r}{R}\right) \quad \text{for} \quad r_1 \leqslant r \leqslant R. \tag{6}$$

Flow patterns at different Rossby numbers were presented there for specified values of r_1 and Q, which describe the shape of the nozzle.

The drag D can be obtained by computing the differences in pressure and axial momentum between a far upstream and a far downstream section. Upon use of the stream functions (1) and (2), we obtain the drag coefficient

$$C_D = \sum_{n=1}^{N} [b_n k_n R \lambda_n J_0(\lambda_n)]^2, \tag{7}$$

which is defined as $D/\frac{1}{2}\rho W^2 \pi R^2$. The expression is equivalent to that derived by Fraenkel.

It shows that there is no drag when $R_o^{-1} \leqslant \lambda_1$, since $b_n = 0$ and waves do not appear in the flow at such low swirl ratios. When R_o^{-1} exceeds the value λ_1 the drag becomes finite, and we like to compare C_D at different swirl ratios for a given nozzle. However, in using this inverse method it is impossible to generate the same nozzle shape, described by $\omega_{\pm}/\frac{1}{2}WR^2 = 1$, at different Rossby numbers. It has been shown by the previous paper that the nozzle is symmetric about $z = 0$ when $R_o^{-1} \leqslant \lambda_1$, and becomes more and more non-symmetric when R_o^{-1} successively exceeds the λ_n's. In the present work we will fix the throat radius at $0{\cdot}9\,R$ and choose arbitrarily a contour shape at a swirl ratio within the range $\lambda_1 < R_o^{-1} < \lambda_2$ as our reference. At any other ratio we adjust the values of r_1 and Q until the nozzle has the same throat radius and its shape becomes as close as possible to the reference one.

When R_o^{-1} approaches a λ_n closely from below, the contour becomes elongated and deviates greatly from the desired shape, and the results in these regions will be omitted. When $R_o^{-1} = \lambda_n$ the flows up- and downstream from the throat are blocked. Under such a critical condition velocity distribution becomes identical along the tube, waves disappear, and wave drag vanishes. A slight increase in R_o^{-1} over the value λ_n gives a regular nozzle shape and a finite drag coefficient.

3. Results

The mean nozzle contours for $\lambda_n < R_o^{-1} < \lambda_{n+1}$, where $n = 1, 2$ and 3, are plotted in figure 1. Within each range of R_o^{-1} there are small deviations about the mean shape. Increasing in R_o^{-1} causes the contour to lean more toward the upstream direction.

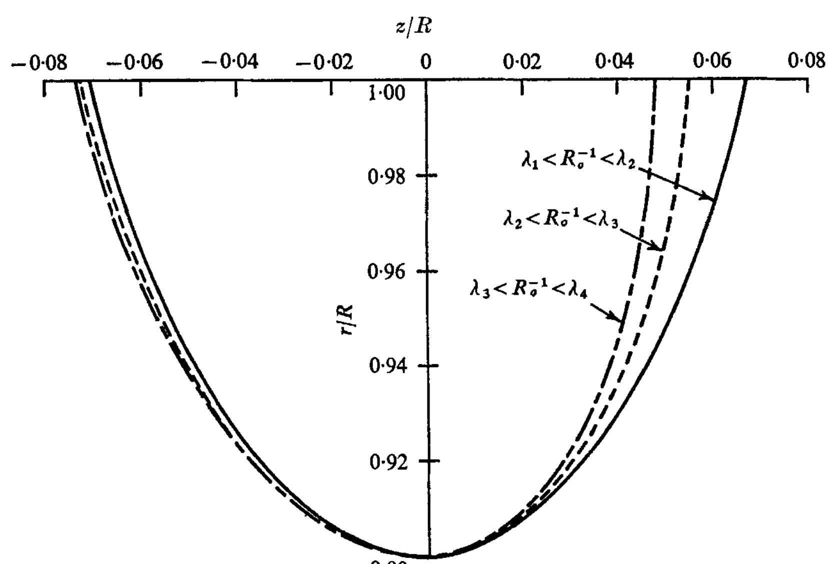

FIGURE 1. The mean nozzle contours for three regions of the swirl ratio.

Figure 2 shows the plot of drag coefficient as a function of R_o^{-1}. It jumps from zero to a finite value when R_o^{-1} passes the first critical value λ_1, and decreases continuously with further increase in R_o^{-1}. After the value λ_2 is passed, the drag jumps from zero to a value which is one order higher than those computed in the previous range of the swirl ratio. A jump occurs again when R_o^{-1} exceeds λ_3 to bring the magnitude of C_D to another higher order.

Concluded from the results obtained so far, we can expect that for still higher values of R_o^{-1}, the drag coefficient will behave in a similar way, that is, its magnitude jumps to a higher order after R_o^{-1} exceeds each critical value, and then tapers off toward zero. Thus at a large swirling ratio the nozzle may experience a wave drag which is far greater than the drag caused by viscosity.

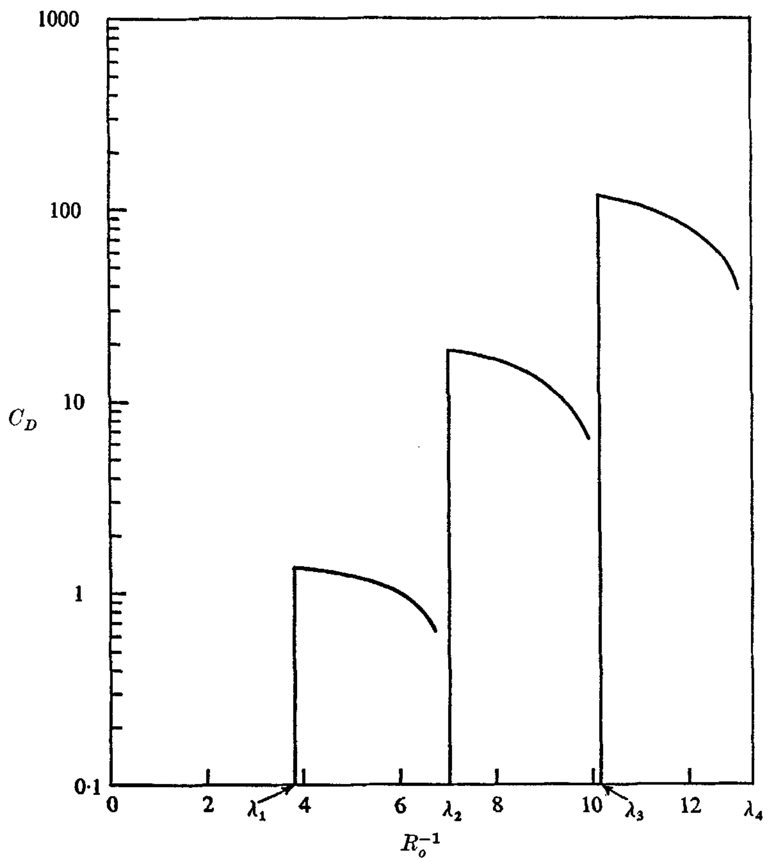

FIGURE 2. The wave drag coefficient as a function of swirl ratio.

From the similarities between a rotating flow and a flow of a stratified fluid, we can anticipate that the drag caused by lee waves will have similar behaviour to that discussed here.

This work was supported by the Advanced Research Projects Agency, contract DA 31-124-ARO-D-139, and the Office of Naval Research, contract nonr 1147(10).

REFERENCES

CHOW, C-Y. 1969 Swirling flow in tubes of non-uniform cross-sections. *J. Fluid Mech.* **38**, 843.

FRAENKEL, L. E. 1956 On the flow of rotating fluid past bodies in a pipe. *Proc. Roy. Soc.* A **233**, 506.